Lecture 14: Convergence & Completeness, Bases,

- Recall that a sequence $\{x_n\}$ converges to x, written as $x_n \to x$, if
  $$\lim_{n\to\infty} x_n = x$$
  or, equivalently, for all $\varepsilon > 0$ there exists $N_0 \in \mathbb{N}$ so that if $n > N_0$, $|x_n - x| < \varepsilon$.

- In a vector space, we replace the absolute value by a norm: $\|x_n - x\| < \varepsilon$. Thus, we have convergence defined in any normed vector space.

- Recall that $L^p(\Omega)$, and $C_c^\infty(\Omega) \subseteq L^p(\Omega)$ are vector spaces. We often use smooth functions to motivate our goals. We now formalize this:

$\boxed{\text{Thm } 7.5}$ Assume $1 \le p < \infty$. For $f \in L^p(\Omega)$, there exists a sequence $\{\psi_k\} \subseteq C_c^\infty(\Omega)$ such that
$$\lim_{k\to\infty} \|\psi_k - f\|_p = 0.$$

$\boxed{\text{Pf}}$ Beyond this course - uses mollifiers. $\square$

-) In otherwords, C_c^∞ is dense in L^p. We use this fact to create a sequence of approximate solutions to PDE's & show convergence.

-) We usually don't know the limit, so we can't show $\|\psi_k - f\| \to 0$ directly. Instead, we use the fact that L^p is <u>complete</u> and show that $\{\psi_k\}$ is a Cauchy sequence:

  $\{v_k\} \subseteq V$ is called Cauchy if for all $\varepsilon > 0$ there exists $N \in \mathbb{N}$ so for $k, m > N$,
  $$\|v_k - v_m\| < \varepsilon.$$

- This establishes the limit.

★ A complete space is one in which every Cauchy sequence has a limit.

•) Notice that

$$\| v_{n_k} - v_m \| \le \| v_{n_k} - v \| + \| v - v_m \|$$

by the triangle inequality, so every convergent sequence is Cauchy. In a complete space, the converse is also true.

ex.) 1.) Notice $1/n \to 0$. One may check directly

$$\| 1/n - 1/m \| \le \frac{1}{nm} \cdot |n-m|$$

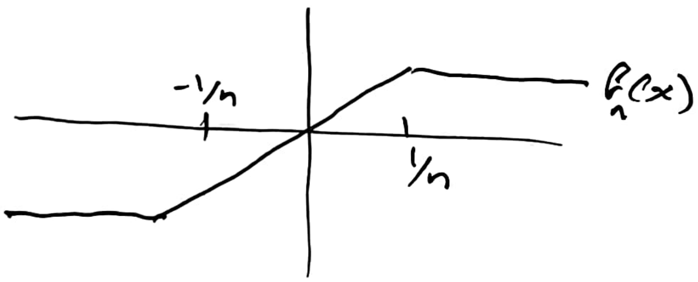
WLOG, let $n > m$ and

$$\| 1/n - 1/m \| \le \frac{1}{nm} \cdot 2n \le 2/m.$$

If $n, m \ge N$, then $\| 1/n - 1/m \| \le 2/N$ and so this sequence is Cauchy.

2.) Consider $C^0([-1,1])$ with the $\mathcal{L}^1$ norm. For $n \in \mathbb{N}$,

$$f_n(x) = \begin{cases} -1 & x < -1/n \\ nx & -1/n \le x \le 1/n \\ 1 & x > 1/n \end{cases}$$



Notice $\| f_n - f_m \|_1 = \int_{-1}^{1} | f_n - f_m | dx = | 1/n - 1/m |$

such that $\{f_n\}$ is Cauchy.

Let $f = \begin{cases} 1 & x > 0 \\ -1 & x < 0 \end{cases}$ and $\lim_{n \to \infty} \| f_n - f \|_1 = 0$

•) We approach Lebesgue theory because it gives convergence results

$\boxed{\text{Th}^{\underline{m}} \quad 7.7}$ For $\Omega \subseteq \mathbb{R}^n$, $\mathcal{L}^p(\Omega)$ is complete under the p-norm.

$\boxed{\text{Pf}}$ Beyond this class - uses the ~~Monotone~~ ~~dominated~~ Monotone + Dominated convergence thm. $\square$

• In functional analysis, a complete normed vector space is called a Banach space.

• A subspace $W \subset V$ is called closed if it is closed in the norm topology: If $\{x_n\} \subseteq W$ converges to $x \in V$, then $x \in W$.

$\boxed{\text{Lemma } 7.8}$ If V is a complete normed vector space and $W \subset V$ is a closed subspace, then W is complete with respect to the norm of V.

$\boxed{\text{Pf}}$ Let $\{x_n\} \subseteq W$ be cauchy. Since V is complete, $x_n \to x$ in V, so $x_n \to x$ in W as W is closed. $\square$

• The $\mathcal{L}^p$ spaces have discrete analogues. Consider sequences $(a_1, a_2, \dots)$ so $a_i \in \mathbb{C}$. We associate a function $a: \mathbb{N} \to \mathbb{C}$ so $a(j) = a_j$. Then

$$\|a\|_p = \left[\sum_{j=1}^{\infty} |a_j|^p\right]^{1/p}$$

and we track the vector space

$$\ell^p(\mathbb{N}) = \{a: \mathbb{N} \to \mathbb{C} \mid \|a\|_p < \infty\}$$

e.g.) ℓ^1 is the set of absolutely summable sequences,
ℓ^∞ is the set of bdd. sequences.

Orthonormal Bases

- A Hilbert Space is a complete vector space with an inner product e.g. $\mathcal{L}^2(\mathbb{R})$.

- Let $\mathcal{H}$ be an infinite dimensional complex Hilbert space.

  ex.) $C_c^\infty(\mathbb{R};\mathbb{C})$ is infinite-dimensional since we may form smooth bump φ_n so $\text{supp}(\varphi_n) \subseteq [n, n+1]$. Indeed, since $\text{supp}(\varphi_n) \cap \text{supp}(\varphi_m) = \emptyset$ if $n \neq m$, $\langle \varphi_m, \varphi_n \rangle = \int \varphi_m \overline{\varphi_n} dx = 0$ and the φ_n are linearly independent.

- A sequence ~~of~~ of vectors $\{e_i\} \subseteq \mathcal{H}$ is orthonormal if
$$\langle e_j, e_k \rangle = \begin{cases} 1 & j = k \\ 0 & j \neq k \end{cases}$$

- An orthonormal basis for $\mathcal{H}$ is an orthonormal sequence such that ~~for~~ each $v \in \mathcal{H}$ admits a unique representation as a convergent sequence
$$v = \sum_{j=1}^{\infty} v_j e_j \quad \text{for } v_j \in \mathbb{C}$$
Note: $v_j = \langle v, e_j \rangle$

  example:) ℓ^2 has orthonormal basis
$$e_i = (0, 0, \ldots, 0, \underset{i\text{th spot.}}{1}, 0, \ldots)$$

$\leadsto$ We often use eigenfunctions of some operator to form an orthonormal basis. Thus, we try to show that partial sums $S_n[v] = \sum_{j=1}^{n} v_j e_j$ converge to v in $\mathcal{H}$ for every $v \in \mathcal{H}$.

$\boxed{\text{Th}^{\text{m}}\ 7.9}$ Bessel's Inequality

Assume that $\{e_j\}$ is an orthonormal sequence in an infinite-dimensional Hilbert Space $\mathcal{H}$. For $v \in \mathcal{H}$, the series ~~$\Sigma |v_j|$~~ $\Sigma |v_j|^2$ converges and

$$\Sigma_{j=1}^{\infty} |v_j|^2 \leq \|v\|^2 \qquad (\text{for } v_j = \langle e_j, v \rangle)$$

Equality Holds iff. $S_n[v] \to v$ in $\mathcal{H}$.

$\boxed{\text{Pf}}$ $\|v - S_n[v]\|^2 = \langle v - S_n[v], v - S_n[v] \rangle$
$= \|v\|^2 - 2\,\mathrm{Re}(\langle S_n[v], v \rangle) + \|S_n[v]\|^2$

→ Since $\{e_j\}$ is orthonormal,
$$\langle S_n[v], v \rangle = \langle S_n[v], S_n[v] \rangle = \Sigma_{j=1}^{n} |v_j|^2$$

$\Rightarrow \|v - S_n[v]\|^2 = \|v\|^2 - \Sigma_{j=1}^{n} |v_j|^2$

or $0 \leq \|v\|^2 - \Sigma_{j=1}^{n} |v_j|^2$

as $n \to \infty$, $\Sigma_{j=1}^{\infty} |v_j|^2 \leq \|v\|^2$ with equality iff $\|v - S_n[v]\|^2 \to 0$. $\square$

Remark: Why can we pass an infinite sum through the inner product? By Cauchy's inequality
$\langle v, w \rangle \leq \|v\| \cdot \|w\|$,
the map $w \mapsto \langle w, S_n[v] \rangle$ is continuous in $\mathcal{H}$

So
$$\lim_{m \to \infty} \langle S_m[v], S_n[v] \rangle = \langle v, S_n[v] \rangle$$
$$\| \qquad$$
$$\langle S_n[v], S_n[v] \rangle.$$

$\boxed{\text{Th}^{\text{m}}\ 7.10}$ Suppose $\mathcal{H}$ is an infinite-dimensional Hilbert space.

An orthonormal Sequence $\{e_j\}$ is a basis if and only if $0 \in \mathcal{H}$ is the only element in $\mathcal{H}$ that is orthogonal to all vectors in the sequence.

$\boxed{\text{Pf}}$ First, let $\{e_j\}$ be a basis. Then, $\langle v, e_j \rangle = 0$ for all j means $v = \Sigma\, 0 \cdot e_j = 0$.

Second, let the orthonormal sequence $\{e_j\}$ satisfy the given property.

For $v \in \mathcal{H}$,

$\sum_{j=1}^{\infty} |v_j|^2 \leq \|v\|^2 < \infty$ such that $\sum_{j=1}^{\infty} v_j e_j = w$ is a point in $\mathcal{H}$.

Consider $y = v - w$. Then, $\langle y, e_j \rangle = v_j - v_j = 0$

So $y \perp e_j$ for all j and so $y = 0$, $v = w$.